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Resonances and Time Reversal Operator in Rigged Hilbert Spaces

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We present a formulation of time reversal for quantum systems with resonances. This interpretation follows with the idea that these systems are intrinsically irreversible. This formulation is made in terms of rigged Hilbert spaces.

1. INTRODUCTION

This paper discusses the role of time reversal in physical systems with resonances. As these systems are commonly presented as examples of irreversible systems [26–29, 2, 4, 5], it is of interest to study them in relation to time reversal.

Based on the work of Wigner [32, 33], Bohm has presented a rather ellaborate theory on this subject [11, 14]. He deals with nonrelativistic quantum resonances produced by quantum scattering. The aim of the present paper is to discuss some aspects of the work of Bohm with mathematical rigor and to complete it.

Resonant scattering is a quantum scattering process which produces resonances. This is extensively discussed in standard books of scattering theory and quantum mechanics as well as in many papers [6, 25, 8–11, 13, 5, 2].

In resonant scattering, resonances are usually defined, in the energy representation, as pairs of poles of the analytic continuation of the S-matrix to the second sheet of a two-sheeted Riemann surface [6]. This description is derived from causality. However, in contrast to what is usual in quantum mechanics, it is not formulated in terms of state vectors. This is very natural

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when we describe elementary particles. In fact, decaying particles are elementary particles. They only differ from stable particles in the value of a parameter: the inverse of the mean lifetime. This is zero for stable particles and nonzero for decaying ones. Thus, decaying particles should be properly described by a wave function in the same sense as stable particles are.

However, most decaying particles have a decay mode which is purely exponential. A purely exponential decaying vector must have a Breit–Wigner energy distribution which is nonzero for all values of the energy [6, 16]. This is incompatible with the semiboundedness of the Hamiltonians. Therefore, no vector in Hilbert space can describe an exponentially decaying state.

There have been, however, several attempts to construct such wave functions as vector states. These vector states are usually called *Gamow vectors*, honoring G. Gamow, who make the first attempt toward a consistent definition of them [18]. A good state vector describing an exponentially decaying particle should fulfill two conditions:

(i) It should be an eigenvector of the total Hamiltonian (the Hamiltonian describing the interaction in the scattering process) with a complex eigenvalue, which coincides with the S-matrix pole. Its real part should be the resonance energy E_R and its imaginary part the half-width $\Gamma/2$ (twice the inverse of the mean lifetime). This again shows that this state cannot belong to a Hilbert space, since Hamiltonians on Hilbert space are self-adjoint operators that have real eigenvalues only.

(ii) It should show an exponential decay time behavior (or exponential growth if we are speaking about the process of creation of a resonance) starting from the instant in which the resonance has been prepared and starts to decay.

It was realized some time ago by Bohm [7] that both conditions can be fulfilled if instead of using Hilbert spaces as a framework for quantum mechanics, we use rigged Hilbert spaces (RHS). RHS have been used in order to give mathematical rigor to the Dirac formulation of quantum mechanics. The need for this kind of structure in quantum mechanics came originally from the postulate of Dirac that any observable has a complete set of eigenvectors. This postulate cannot be implemented in Hilbert space (many observables have no eigenvectors in Hilbert space, like the momentum and position operators), but it can in the RHS idealization of QM, through a well-known theorem due to Gelfand and Maurin.

The definition and properties of RHS have been presented many times [19, 20, 7, 30, 23, 1, 24]. The construction of Gamow vectors for single- or multiple-pole resonances (resonances described by a pair of simple or multiple poles of the S-matrix, respectively) can be found in ref. 17 and the references quoted therein and here.

According to previous work, resonance scattering processes can be described by two RHS:

$$\Phi^{+} \subset \mathcal{H}_{ac} \subset {}^{\times}\Phi^{+} \tag{1.1}$$

for the process of formation of a resonance. This process takes place at negative values of time, where the origin of time is conventionally settled as the instant at which the process of formation of a resonance is completed and starts to decay.

The second RHS describes the process of decay itself and is given by

$$\Phi^{-} \subset \mathcal{H}_{ac} \subset {}^{\times}\Phi^{-} \tag{1.2}$$

In both RHS the Hilbert space $\mathcal{H}_{ac}$ represents the space of scattering states of the total Hamiltonian H. In fact, "ac" stands for absolutely continuous. On $\mathcal{H}_{ac}$, H has absolutely continuous spectrum only.

Based on the work of Ludwig [21, 22], Bohm gave a further interpretation to these RHS [6]. Although the state vectors in $\Phi^{\pm}$ evolve with the interaction dynamics given by H, they have a close connection to free state vectors through the Möller operators. In this sense, state vectors in Φ^{+} come from free vectors that have been prepared as free before the interaction took place. Analogously, state vectors in Φ^{-} will be observed as free after the interaction takes place. We can prepare incoming states, but after scattering we observe the outgoing states. Thus, it seems natural to assume that the RHS (1.1) gives the set of *preparable* states and (1.2) the set of observables. In this sense, the concept of state is prior to the concept of observable. This implies the existence of an arrow of time [12].

As we shall see in Section 2, the standard time reversal operator transforms the RHS (1.1) into (1.2) and vice versa. If we accept the above interpretation, this would imply that the time reversal operator transforms states into observables and vice versa, which is unacceptable, since it should transform states into states and observables into observables.

Due to this circumstance, Bohm pointed out that for a proper description of the time reversal operation for irreversible systems like quantum systems with resonances, the standard time reversal operator is not sufficient [11]. However, Wigner proposed four types of time reversal operator based on the study of the representations of the Poincaré group extended by inversions [32, 33]. The first type is the standard one, but the other three imply a doubling of spaces. The "good" time reversal operator is one of the latter, and therefore a doubling of spaces in our formalism for resonances is needed. One of the goals of the present paper is to make a mathematical presentation of this doubling.

Now, let us recall very briefly some of the properties of the (single-pole) Gamow vectors and assume that the analytic continuation of the S

matrix has a simple pole (on the second sheet) at the points $z_R = E_R - i\Gamma/2$ and $z_R^* = E_R + i\Gamma/2$. Following refs. 17 and 3, we denote the Gamow vectors corresponding to this resonance as $|f_0\rangle$ and $|\tilde{f}_0\rangle$. They have the following properties [17]:

(i) The Gamow vectors are functionals:

$$|f_0\rangle \in {}^{\times}\Phi^{-} \qquad \text{and} \qquad |\tilde{f}_0\rangle \in {}^{\times}\Phi^{+} \tag{1.3}$$

(ii) They are generalized eigenvectors of the total Hamiltonian H (for a definition of generalized eigenvector and eigenvalue, see ref. 12):

$$H|f_0\rangle = z_R|f_0\rangle; \qquad H|\tilde{f}_0\rangle = z_R^*|\tilde{f}_0\rangle \tag{1.4}$$

(iii) The time evolution operator e^{-itH} can be continuously extended to ${}^{\times}\Phi^{-}$ for *positive* values of time and to ${}^{\times}\Phi^{+}$ for *negative* values of time. The continuity of the extensions refers to the weak topology [31]. In addition, we have

$$e^{-itH}|f_0\rangle = e^{-itz_R}|f_0\rangle = e^{-itE}e^{-\Gamma t}|f_0\rangle \qquad \text{for} \quad t > 0 \tag{1.5}$$

and

$$e^{-itH}|\tilde{f}_0\rangle = e^{-itz_R^*}|\tilde{f}_0\rangle = e^{-itE}e^{\Gamma t}|\tilde{f}_0\rangle \qquad \text{for} \quad t < 0 \tag{1.6}$$

The action of e^{-itH} on $|f_0\rangle$ for $t < 0$ and on $|\tilde{f}_0\rangle$ for $t > 0$ is, however, not defined.

From this latter property, we can see that the group giving the evolution of states and observables splits into two semigroups. This splitting is a consequence of the choice of $\Phi^{\pm}$ and the properties of Hardy functions. The choice of Hardy functions is related to a causality condition [12], so that the splitting is also related to causality. The need for two RHS is due to the existence of two different processes in resonance scattering: formation and decay. The process of formation takes place in the past, the process of decay in the future. This formulation implies the existence of an arrow of time [12]. As we shall see in Section 2, formation and decay are related through the time reversal operator. As discussed by several authors [26–29, 2, 5, 12], this splitting shows that scattering processes with resonances are *irreversible processes.*

The present paper is divided in three following sections and one appendix. In the next section, we discuss the effect of time reversal on Gamow vectors in the case of single-pole resonances. In Section 3 we present the idea of *doubling* of spaces. The discussion for the more general multiple-pole resonances is left for Section 4. In the Appendix we present some general aspects of the time reversal operation.

2. SINGLE-POLE RESONANCES: THE STANDARD CASE

In the present section we present the effect of the time reversal operation for systems having single-pole resonances, which is the most common case. This means that the poles of the S-matrix at the points z_R and z_R^* are simple.

All the concepts used here are presented in the standard literature on the subject, in particular in refs. 6, 13, and 17. The notation does not differ essentially from that in ref. 17, although there exist a couple of differences:

(i) The restriction of the spaces, which are intersections of Hardy spaces with the Schwartz space, to the positive semiaxis is here denoted by

$$\mathcal{H}^2_\pm \cap S\big|_{\mathbb{R}^+}$$

The plus sign stands for Hardy functions on the upper half-plane and the minus sign for Hardy functions on the lower half-plane. S is the Schwartz space.

(ii) The extensions of the evolution operator e^{-itH} are called $\mathcal{U}_-(t)$ if $t > 0$ and $\mathcal{U}_+(t)$ if $t < 0$. These are the two semigroups discussed in the Introduction. The former is defined and continuous on ${}^\times\Phi^-$ and the latter on ${}^\times\Phi^+$.

(iii) For simplicity, we always work with a spherically symmetric potential and have particles without spin and other possible degress of freedom. We restrict ourselves to a zero value of the angular momentum and we represent the Hilbert space of states $\mathcal{H}_0$ (for $l = 0$). In this context, the space of our scattering states $\mathcal{H}_{ac}$ is a subspace of $\mathcal{H}_0$.

Let us recall that the unitary operators given by $V_\pm$ *diagonalize* the total Hamiltonian H (or its restriction to its absolutely continuous space $\mathcal{H}_{ac}$, if H has bound states or continuous singular spectrum), in the sense that these operators give a unitary equivalence between H and the multiplication operator on $L^2(\mathbb{R}^+)$. They are a product of the inverses of the Møller operators times the operator U that *diagonalizes* the free Hamiltonian, $V_\pm = \Omega_\pm^{-1}U$.

Standard Case. For s waves ($j = 0$ and for particles without spin $\sigma = 0$), the simplest choice for the time reversal operation is $\epsilon_T = \epsilon_I = 1$, and therefore $A_T = C$ in the energy representation. Since the mapping C transforms any function on E into its complex conjugate, it maps

$$C: \quad \mathcal{H}^2_\pm \cap S\big|_{\mathbb{R}^+} \mapsto \mathcal{H}^2_\mp \cap S\big|_{\mathbb{R}^+} \tag{2.1}$$

Moreover, we can show that this map is continuous.

Our next goal is to define time reversal operators $A_{T\pm}$ on $\Phi^\pm$. These operators should be equivalent to C and the equivalence should be given by $V_\pm$. Their definition is

$$A_{T\pm} := V_\mp^\dagger C V_\pm \tag{2.2}$$

This definition makes commutative the following diagram:

$$\begin{array}{ccc} \mathcal{H}^2_\pm \cap S\big|_{\mathbb{R}^+} & \xrightarrow{\;C\;} & \mathcal{H}^2_\mp \cap S\big|_{\mathbb{R}^+} \\ {\scriptstyle V_\mp^\dagger = V_\mp^{-1}}\Big\downarrow & & \Big\downarrow{\scriptstyle V_\mp^\dagger = V_\pm^{-1}} \\ \mathbf{\Phi}^\mp & \xrightarrow[A_{T\mp}]{} & \mathbf{\Phi}^\pm \end{array}$$

These operators have the following properties:

1. $A_{T\pm}$ are continuous antilinear mappings from $\mathbf{\Phi}^\pm$ onto $\mathbf{\Phi}^\mp$.

2. They can be extended to (continuous) antiunitary mappings from $\mathcal{H}_0$ into itself (or $\mathcal{H}_{ac}$ if H has eigenvectors in $\mathcal{H}_0$).

3. Their adjoints are given by

$$A^\dagger_{T\pm} = [V^\dagger_\mp C V_\pm]^\dagger = V^\dagger_\pm C V_\mp = A_{T\mp} \tag{2.4}$$

so that they are adjoints of each other.

4. They are inverses of each other:

$$T_{T+} A_{T-} = V^\dagger_- C V_+ V^\dagger_+ C V_- = I \qquad \text{on} \quad \mathbf{\Phi}^- \tag{2.5}$$

Analogously, $A_{T-} A_{T+} = I$ on $\mathbf{\Phi}^+$.

Now, let us assume that $\mathbf{A}$ is a densely defined, continuous antilinear operator on $\mathcal{H}$ with the following property: There are two RHS, $\mathbf{\Phi} \subset \mathcal{H} \subset {}^\times\mathbf{\Phi}$ and $\mathbf{\Psi} \subset \mathcal{H} \subset {}^\times\mathbf{\Psi}$, such that $A^\dagger$ maps continuously $\mathbf{\Psi}$ into $\mathbf{\Phi}$. Then, A can be extended by continuity to ${}^\times\mathbf{\Phi}$ using the following formula:

$$\langle A^\dagger\psi \big| F\rangle = \langle AF \big| \psi\rangle; \qquad \forall F \in {}^\times\mathbf{\Phi}, \quad \forall \psi \in \mathbf{\Psi} \tag{2.6}$$

Thus, A is a weakly continuous mapping from $\mathbf{\Phi}^\times$ into $\mathbf{\Psi}^\times$. The proof is this result goes exactly as the proof for the linear case, which is given in any book on topological vector spaces.

It is quite simple to apply these ideas to our particular case, after making the identification $\mathbf{\Phi} = \mathbf{\Phi}^\pm$, $\mathbf{\Psi} = \mathbf{\Phi}^\mp$, $A = A_{T\pm}$, and $A^\dagger = A_{T\mp}$. Thus, we have the following continuous antilinear extensions:

$$A_{T\pm}\colon \quad {}^\times(\mathbf{\Phi}^\pm) \mapsto {}^\times(\mathbf{\Phi}^\mp) \tag{2.7}$$

They are one-to-one onto mappings with continuous inverses that indeed extend $A_{T\pm}$ as originally defined in (2.2). For simplicity, we assign the same notation to these operators and their respective extensions.

It is interesting to find the images of $\big|E^\pm\rangle$ and the Gamow vectors by $A_{T\pm}$. To this end, let us consider two arbitrary vectors $\varphi^\pm \in \mathbf{\Phi}^\pm$. Their wave functions in the energy representation are given by

$$\varphi^{\mp}(E) = \langle E^{\pm}|\varphi^{\pm}\rangle = (V_{\pm}\varphi^{\pm})(E) \in \mathcal{H}^2_{\mp} \cap S|_{\mathbb{R}^+} \tag{2.8}$$

Using the definition of $A_{T\pm}$, we have

$$[A_{T\pm}\varphi^{\pm}](E) = [V_{\mp}A_{T\pm}\varphi^{\pm}](E) = C[V_{\pm}\varphi^{\pm}](E) = C\varphi^{\mp}(E) = [\varphi^{\mp}(E)]^* \tag{2.9}$$

Hence,

$$\langle E^{\mp}|A_{T\pm}\varphi^{\pm}\rangle = (A_{T\pm}\varphi^{\pm})(E) = [\varphi^{\mp}(E)]^* = \langle\varphi^{\pm}|E^{\pm}\rangle \tag{2.10}$$

According to (2.4) and (2.6), we finally obtain

$$\langle E^{\mp}|A_{T\pm}\varphi^{\pm}\rangle = \langle\varphi^{\pm}|A_{T\mp}|E^{\mp}\rangle = \langle\varphi^{\pm}|E^{\pm}\rangle \tag{2.11}$$

and then

$$A_{T\pm}|E^{\pm}\rangle = |E^{\mp}\rangle \tag{2.12}$$

Take now the Gamow vectors $|z_R^-\rangle = |f_0\rangle$ and $|z_R^{*+}\rangle = |\tilde{f}_0\rangle$. From (2.6) one has

$$\begin{aligned}\langle\varphi^+|A_{T-}|f_0\rangle &= \langle f_0|A_{T+}\varphi^+\rangle = (A_{T+}\varphi^+)(z_R) = [\varphi^+(z_R^*)]^* \\ &= \langle\tilde{f}_0|\varphi^+\rangle^* = \langle\varphi^+|\tilde{f}_0\rangle\end{aligned} \tag{2.13}$$

Since (2.13) is valid for any $\varphi^+ \in \Phi^+$, one finally obtains

$$A_{T-}|f_0\rangle = |\tilde{f}_0\rangle \tag{2.14}$$

Analogously

$$A_{T+}|\tilde{f}_0\rangle = |f_0\rangle \tag{2.15}$$

Next, we show a property that is related to the semigroup splitting: take e^{iHt} with $t > 0$. We know that $e^{iHt}\Phi^- \subset \Phi^-$. Analogously, if $t < 0$, we have $e^{iHt}\Phi^+ \subset \Phi^+$. Now, consider $(t > 0)$

$$\begin{aligned}A_{T+}e^{iHt}A_{T-} &= V_-^{\dagger}CV_+e^{iHt}V_+^{\dagger}CV_- \\ &= V_-^{\dagger}Ce^{itE}CV_- = V_-^{\dagger}e^{-itE}V_- = e^{i(-t)H} \qquad (t > 0)\end{aligned} \tag{2.16}$$

For $t < 0$ we get

$$A_{T-}e^{iHt}A_{T+} = e^{i(-t)H} \tag{2.17}$$

We see that the operators $A_{T\pm}$ transform one semigroup into the other. After the above formulas, we can extend these formulas to the duals by duality [13, 17]. If $t > 0$

$$A_{T+}\mathcal{U}_-(t)A_{T-} = \mathcal{U}_+(-t) \tag{2.18}$$

and if $t < 0$

$$A_{T-}\mathcal{U}_+(t)A_{T+} = \mathcal{U}_-(-t) \tag{2.19}$$

Remark. We could expect that the operators $A_{T\pm}$ are indeed the same in some sense. That is true. As a matter of fact, their extensions to $\mathcal{H}_0$ (or $\mathcal{H}_{ac}$) coincide. The proof of this statement is rather simple. Write

$$A_{T+}^2 = V_-^\dagger C V_+ V_-^\dagger C V_+ \tag{2.20}$$

Since $V_\pm = U\Omega_\pm^\dagger$, one has

$$A_{T+}^2 = \Omega_- U^\dagger C U \Omega_+^\dagger \Omega_- U^\dagger C U \Omega_+^\dagger \tag{2.21}$$

where $\Omega_\pm$ are the Møller operators and U diagonalizes the free Hamiltonian (see the beginning of this section). The S operator is $S = \Omega_-^\dagger \Omega_+$, with

$$USU^{-1} = USU^\dagger = S(E) \qquad [= S(E + i0), \quad E > 0] \tag{2.22}$$

$$S^\dagger = \Omega_+^\dagger \Omega_- \Rightarrow US^\dagger U^\dagger = (USU^\dagger)^\dagger = S^*(E) \tag{2.23}$$

Therefore,

$$\begin{aligned} A_{T+}^2 &= \Omega_- U^\dagger C S^*(E) C U \Omega_+^\dagger \\ &= \Omega_- U^\dagger S(E) U \Omega_+^\dagger = \Omega_- S \Omega_+^\dagger \\ &= \Omega_- \Omega_-^\dagger \Omega_+ \Omega_+^\dagger = I \end{aligned} \tag{2.24}$$

which is the identity on $\mathcal{H}_0$ or $\mathcal{H}_{ac}$. The same is true for A_{T-}.

We have seen that A_{T+} is an inversible bounded operator such that $A_{T+} = A_{T+}^{-1}$ on $\mathcal{H}_0$ or $\mathcal{H}_{ac}$. Since A_{T+}^{-1} and A_{T-} coincide on the dense subspace Φ^-, they are equal on this Hilbert space and we have that $A_{T-} = A_{T+}^{-1} = A_{T+}$.

3. SINGLE-POLE RESONANCES: THE CASE $\epsilon_T = \epsilon_I = 1$ (*s* WAVES)

From the point of view of Ludwig [21, 22] and Bohm [11, 12, 14], $A_{T\pm}$ are not good time reversal operators because they transform preparable objects into observable objects and vice versa. In other words, they transform states into observables and vice versa. In order to avoid this difficulty, Bohm [11, 14] suggested using another definition of the time reversal operation. Following Wigner [32, 32], there are three other possibilities of defining the time reversal operator. All four choices are listed in the Appendix. Bohm uses the fourth in this list, because it does not reverse parity. Based on refs.

11 and 14, we present a mathematical formulation of time reversal with space doubling.

To begin with, let us consider the following pair of RHS:

$$\mathcal{H}^2_{\pm} \cap S\big|_{\mathbb{R}^+} \otimes \mathbb{C}^2 \subset L^2(\mathbb{R}^+) \otimes \mathbb{C}^2 \subset {}^{\times}(\mathcal{H}^2_{\pm} \cap S\big|_{\mathbb{R}^+}) \otimes \mathbb{C}^2 \tag{3.1}$$

where $\mathbb{C}^2$ denotes the two-dimensional vector space of column vectors whose entries are complex numbers. Then the elements of each space of the triplet can be expressed as two-dimensional vectors whose entries belong to the space in the left-hand side of the tensor product. Now, the time reversal operator in the energy representation is given by

$$\mathcal{C} := \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix} \tag{3.2}$$

This operator is antilinear and continuous from $\mathcal{H}^2_{\pm} \cap S\big|_{\mathbb{R}^+} \otimes \mathbb{C}^2$ into $\mathcal{H}^2_{\mp} \cap S\big|_{\mathbb{R}^+} \otimes \mathbb{C}^2$, since C represents complex conjugation. By duality, it can be extended to a continuous antilinear mapping from ${}^{\times}(\mathcal{H}^2_{\pm} \cap S\big|_{\mathbb{R}^+}) \otimes \mathbb{C}^2$ onto ${}^{\times}(\mathcal{H}^2_{\mp} \cap S\big|_{\mathbb{R}^+}) \otimes \mathbb{C}^2$.

The spaces $\mathcal{H}^2_{\pm} \cap S\big|_{\mathbb{R}^+} \otimes \mathbb{C}^2$ each have two distinguished subspaces, which are

$$\Sigma^+_{\pm} = \mathcal{H}^2_{\pm} \cap S\big|_{\mathbb{R}^+} \otimes \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \quad \text{and} \quad \Sigma^-_{\pm} = \mathcal{H}^2_{\pm} \cap S\big|_{\mathbb{R}^+} \otimes \begin{pmatrix} 0 \\ \beta \end{pmatrix} \tag{3.3}$$

where α and β are arbitrary complex numbers. We have, therefore, two new RHS that can be written in the following form:

$$\Sigma^{\pm}_{\pm} \subset L^{2\pm}(\mathbb{R}^+) \subset {}^{\times}(\Sigma^{\pm}_{\pm}) \tag{3.4}$$

where

$$L^{2+}(\mathbb{R}^+) = L^2(\mathbb{R}^+) \otimes \begin{pmatrix} \alpha \\ 0 \end{pmatrix}; \qquad L^{2-}(\mathbb{R}^+) = L^2(\mathbb{R}^+) \otimes \begin{pmatrix} 0 \\ \beta \end{pmatrix} \tag{3.5}$$

One can easily prove that the duals can be written in the form

$${}^{\times}(\Sigma^+_{\pm}) = {}^{\times}(\mathcal{H}^2_{\pm} \cap S\big|_{\mathbb{R}^+}) \otimes \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \quad \text{and} \quad {}^{\times}(\Sigma^-_{\pm}) = {}^{\times}(\mathcal{H}^2_{\pm} \cap S\big|_{\mathbb{R}^+}) \otimes \begin{pmatrix} 0 \\ \beta \end{pmatrix} \tag{3.6}$$

Also, it is not difficult to show the following property:

$$\mathcal{C}\Sigma^{\pm}_{\pm} = \Sigma^{\mp}_{\mp} \tag{3.7}$$

Needless to say, $\mathscr{C}$ in the above formula represent a continuous antilinear bijective (one-to-one and onto) mapping between the corresponding spaces. Therefore, it can be continuously extended by duality and its action on the duals is given by

$$\mathscr{C}\ ^{\times}(\Sigma_{\pm}^{\pm}) = {}^{\times}(\Sigma_{\mp}^{\mp}) \tag{3.8}$$

Remark. Note that $\mathscr{C}^2 = -\mathbb{I} = \epsilon_T\mathbb{I}$, where $\mathbb{I}$ represents the identity on ${}^{\times}(\Sigma_{\pm}^{\pm})$.

Now, let us consider the operators given by

$$\mathfrak{D}_{\pm} = V_{\pm} \otimes I \tag{3.9}$$

where I is the identity on $\mathbb{C}^2$. Formally

$$\mathfrak{D}_{\pm} = \begin{pmatrix} V_{\pm} & 0 \\ 0 & V_{\pm} \end{pmatrix}; \qquad \mathfrak{D}_{\pm}^{\dagger} = \mathfrak{D}_{\pm}^{-1} = \begin{pmatrix} V_{\pm}^{\dagger} & 0 \\ 0 & V_{\pm}^{\dagger} \end{pmatrix} \tag{3.10}$$

$\mathfrak{D}_{\pm}$ maps $\mathscr{H}_{\text{ac}} \otimes \mathbb{C}^2$ onto $L^2(\mathbb{R}^+) \otimes \mathbb{C}^2$.

Remark. In order to clarify the notation, we propose to replace the superscript signs by the superscript r with $r = +, -$. Then, one writes, $\Sigma_{\pm}^{r}$, etc. With this notation, we intend to make it clear that the signs above and below are independent.

Let us define the following spaces:

$$\Phi^{\pm,r} := \mathfrak{D}_{\pm}^{\dagger}\Sigma_{\mp}^{r} \tag{3.11}$$

They are subspaces of $\mathscr{H}_{\text{ac}} \otimes \mathbb{C}^2$.

As in the previous cases, $\mathfrak{D}_{\pm}$ and their respective inverses $\mathfrak{D}_{\pm}^{\dagger}$ can be continuously extended by duality to the dual spaces. It is also obvious that

$$\Phi^{\pm,r=+} = \Phi^{\pm} \otimes \begin{pmatrix} \alpha \\ 0 \end{pmatrix}; \qquad \Phi^{\pm,r=-} = \Phi^{\pm} \otimes \begin{pmatrix} 0 \\ \beta \end{pmatrix} \tag{3.12}$$

We are now in a position to introduce the time reversal operators for our particular situation. They can be defined as

$$\mathscr{A}_{T\pm} := \pm\mathfrak{D}_{\mp}^{\dagger}\mathscr{C}\mathfrak{D}_{\pm} \tag{3.13}$$

These two operators have properties that are similar to those of $A_{T\pm}$. We list here these properties without proofs, since these proofs do not differ much from those for $A_{T\pm}$:

1. $A_{T\pm}$ are continuous antilinear mappings from $\Phi^{\pm,r}$ onto $\Phi^{\mp,-r}$, respectively. They can be continuously extended as antilinear mappings between the respective duals.

2. They are adjoints of each other:

$$\mathcal{A}_{T\pm}^{\dagger} = \mathcal{A}_{T\mp} \tag{3.14}$$

3. These operators are defined on the whole space $\mathcal{H}_{\rm ac} \otimes \mathbb{C}^2$ on which they are antiunitary. In addition, they are inverses of each other:

$$\mathcal{A}_{T+}\mathcal{A}_{T-} = \mathbb{I}; \qquad \mathcal{A}_{T-}\mathcal{A}_{T+} = \mathbb{I} \tag{3.15}$$

4. On the Hilbert space $\mathcal{H}_{\rm ac} \otimes \mathbb{C}^2$, we have

$$\mathcal{A}_{T\pm}^{2} = -\mathbb{I} \Rightarrow \mathcal{A}_{T-}\mathcal{A}_{T+}\mathcal{A}_{T+} = -\mathcal{A}_{T-} \Rightarrow \mathcal{A}_{T+} = -\mathcal{A}_{T-} \tag{3.16}$$

This is a consequence of the definition chosen in (3.13) for $\mathcal{A}_{T-}$ (with minus sign), which has its origin in the fact that $\mathcal{C}^{\dagger} = -\mathcal{C}$. If we redefine $\mathcal{A}_{T-}$ without the minus sign in (3.13), we have

$$\mathcal{A}_{T\pm}^{\dagger} = -\mathcal{A}_{T\mp}; \qquad \mathcal{A}_{T+}\mathcal{A}_{T-} = -\mathbb{I}; \qquad \mathcal{A}_{T-}\mathcal{A}_{T+} = -\mathbb{I} \tag{3.17}$$

which yields

$$\mathcal{A}_{T+} = \mathcal{A}_{T-} \tag{3.18}$$

We can choose either the possibility (3.16) or (3.18). The choice (3.18) has the advantage of having a unique time reversal operator. The distinction between $\mathcal{A}_{T+}$ and $\mathcal{A}_{T-}$ indicates the restriction to the unique time reversal operator to $\Phi^{\pm,r}$. This choice has a little inconvenience as it causes the appearance of a minus sign without physical meaning in the time reversal formulas for the semigroups, as we shall see later.

The importance of the above construction lies in the possibility of extending the time reversal operator to the duals in which interesting generalized state vectors live. Such is the case of Gamow vectors, so that the above construction allows us to apply time reversal to these objects. In our particular case, Gamow vectors are slightly more complicated structures than in the standard case studied in the first part of the present section. To begin with, let us consider the state vectors having a definite value of the energy (also called Dirac kets). They are generalized eigenvectors of the total Hamiltonian with positive eigenvalue $E > 0$. In our case, they have the following form:

$$|E^{\pm}; r = +\rangle = \begin{pmatrix} |E^{\pm}\rangle \\ 0 \end{pmatrix} \in {}^{\times}(\Phi^{\pm,r=+}) \tag{3.19}$$

$$|E^{\pm}; r = -\rangle = \begin{pmatrix} 0 \\ |E^{\pm}\rangle \end{pmatrix} \in {}^{\times}(\Phi^{\pm,r=-}) \tag{3.20}$$

Both are generalized eigenvalues of the operator $H \otimes I$ (H is the exact Hamiltonian and I the identity on $\mathbb{C}^2$) with generalized eigenvalue equal to

$E > 0$. We want to determine the action of $\mathscr{A}_{T\pm}$ on these vectors. We start with the following identity, which has its origin in (2.6):

$$\langle \Xi^{\pm} \big| \mathscr{A}_{T\mp} \big| E^{\mp}; r\rangle = \langle E^{\mp}; r \big| \mathscr{A}_{T\pm} \big| \Xi^{\pm}\rangle \qquad \text{for all} \quad \Xi^{\pm} \in \boldsymbol{\Phi}^{\pm,r} \tag{3.21}$$

From (3.13), we obviously obtain

$$\mathscr{A}_{T\pm} = \pm \begin{pmatrix} 0 & A_{T\pm} \\ -A_{T\pm} & 0 \end{pmatrix} \tag{3.22}$$

Let us write

$$\Xi^{\pm} = \begin{pmatrix} \varphi^{\pm} \\ \psi^{\pm} \end{pmatrix} \tag{3.23}$$

where $\varphi^{\pm}, \psi^{\pm} \in \boldsymbol{\Phi}^{\pm}$. We shall study separately two cases: $r = \pm$.

Take $r = +$:

$$\begin{aligned} &\langle E^{\mp}, r = + \big| \mathscr{A}_{T\pm} \big| \Xi^{\pm}\rangle \\ &\quad = \pm(\langle E^{\mp}\big|, 0) \begin{pmatrix} A_{T\pm}\psi^{\pm} \\ -A_{T\pm}\varphi^{\pm} \end{pmatrix} \\ &\quad = \pm\langle E^{\mp}\big| A_{T\pm}\psi^{\pm}\rangle = \pm\langle \psi^{\pm}\big| A_{T\mp}\big| E^{\mp}\rangle \\ &\quad = \pm\langle \psi^{\pm}\big| E^{\pm}\rangle = \pm(\varphi^{\pm}, \psi^{\pm}) \begin{pmatrix} 0 \\ \big|E^{\pm}\rangle \end{pmatrix} = \pm\langle \Xi^{\pm}\big| E^{\pm}, r = -\rangle \end{aligned} \tag{3.24}$$

This chain of identities along (3.2) yields

$$\mathscr{A}_{T\mp}\big|E^{\mp}; r = +\rangle = \pm\big|E^{\pm}; r = -\rangle \tag{3.25}$$

The double sign appears as the coefficient of $\big|E^{\pm}, r = -\rangle$ only if we make the choice $\mathscr{A}_{T-} = -\mathfrak{D}^{\dagger}_{\mp}\mathscr{C}\mathfrak{D}_{\pm}$. The choice $\mathscr{A}_{T-} = \mathfrak{D}^{\dagger}_{\mp}\mathscr{C}\mathfrak{D}_{\pm}$ replaces this double sign in (3.25) by a plus sign.

Now take $r = -$. An analogous calculation gives

$$\mathscr{A}_{T\mp}\big|E^{\mp}; r = -\rangle = \mp\big|E^{\pm}; r = +\rangle \tag{3.26}$$

where the double sign in the coefficient has the same origin. The choice $\mathscr{A}_{T-} = \mathfrak{D}^{\dagger}_{\mp}\mathscr{C}\mathfrak{D}_{\pm}$ replaces it by a minus sign.

The next step is to define the Gamow vectors within this context and obtain their images under time reversal. In a natural way, we have four Gamow vectors:

$$\left|f_0; r = +\right\rangle = \left|z_R^-; r = +\right\rangle = \begin{pmatrix} \left|f_0\right\rangle \\ 0 \end{pmatrix} \tag{3.27}$$

$$\left|f_0; r = -\right\rangle = \left|z_R^-; r = -\right\rangle = \begin{pmatrix} 0 \\ \left|f_0\right\rangle \end{pmatrix} \tag{3.28}$$

$$\left|\tilde{f}_0; r = +\right\rangle = \left|z_R^{*+}; r = +\right\rangle = \begin{pmatrix} \left|\tilde{f}_0\right\rangle \\ 0 \end{pmatrix} \tag{3.29}$$

$$\left|\tilde{f}_0; r = -\right\rangle = \left|z_R^{*+}; r = -\right\rangle = \begin{pmatrix} 0 \\ \left|\tilde{f}_0\right\rangle \end{pmatrix} \tag{3.30}$$

These Gamow vectors are generalized eigenvectors of the operator $H \otimes I$ with eigenvalues z_R for (3.27) and (3.28), and z_R^* for (3.29) and (3.30). We can show the following:

$$\mathcal{A}_{T-}\left|f_0; r = +\right\rangle = -\left|\tilde{f}_0; r = -\right\rangle \tag{3.31}$$

$$\mathcal{A}_{T-}\left|f_0; r = -\right\rangle = \left|\tilde{f}_0; r = +\right\rangle \tag{3.32}$$

$$\mathcal{A}_{T+}\left|\tilde{f}_0; r = +\right\rangle = -\left|f_0; r = -\right\rangle \tag{3.33}$$

$$\mathcal{A}_{T+}\left|\tilde{f}_0; r = -\right\rangle = \left|f_0; r = +\right\rangle \tag{3.34}$$

The overall sign on the right-hand side correspond to the choice $\mathcal{A}_{T-} = +\mathfrak{D}_+^\dagger \mathcal{C} \mathfrak{D}_-$. For the other choice, all the overall signs on the right-hand side must be changed.

Remark. The S operator is written in this context as $S \otimes I$. Therefore, it is a diagonal operator and $S_{u,u'}(E) = \langle u|S(E)|u'\rangle = S(E)\delta_{u,u'}$.

4. MULTIPLE-POLE RESONANCES

4.1. Standard Case

If the analytically continued S-matrtix has a multiple pole of order n at the points $z_R = E_R - i\Gamma/2$ and its complex conjugate z_R^*, new types of Gamow vectors appear [3, 17]. In particular, we have n decaying Gamow vectors $\left|f_0\right\rangle, \left|f_1\right\rangle, \ldots, \left|f_{n-1}\right\rangle \in {}^\times\Phi^-$ and n growing Gamow vectors $\left|\tilde{f}_0\right\rangle, \left|\tilde{f}_1\right\rangle, \ldots, \left|\tilde{f}_{n-1}\right\rangle \in {}^\times\Phi^+$ having the following properties:

(i) The functionals $\left|f_0\right\rangle$ and $\left|\tilde{f}_0\right\rangle$ are the same as in the previous case and have the same properties. In particular, they are generalized eigenvectors of the total Hamiltonian with respective eigenvalues given by z_R and z_R^*. Their time evolution has been discussed in the Introduction.

(ii) For the other functionals, we have

$$H\big|f_k\rangle = z_R\big|f_k\rangle + k\big|f_{k-1}\rangle; \qquad H\big|\tilde{f}_k\rangle = z_R^*\big|\tilde{f}_k\rangle + k\big|\tilde{f}_{k-1}\rangle \tag{4.1}$$

Although these formulas are in principle valid for $k = 1, 2, \ldots, n-1$, we can use them also for $k = 0$, since then their second term on the right-hand side vanishes. Thus, the restriction of H to the subspaces of ${}^{\times}\Phi^{\pm}$ spanned by these Gamow vectors has the typical form of a Jordan block matrix [17].

(iii) Their time evolution is given by

$$e^{-itH}\big|f_k\rangle = \sum_{l=0}^{k}\binom{k}{l}(-it)^{k-l}e^{-itz_R}\big|f_l\rangle \qquad (t > 0) \tag{4.2}$$

and

$$e^{-itH}\big|\tilde{f}_k\rangle = \sum_{l=0}^{k}\binom{k}{l}(-it)^{k-l}e^{-itz_R^*}\big|\tilde{f}_l\rangle \qquad (t < 0) \tag{4.3}$$

In order to obtain the action of the time reversal operators $A_{T\pm}$ on these Gamow vectors, we need to recall some of their properties.

The vectors $\big|f_k\rangle \in {}^{\times}\Phi^{-}$ are functionals on Φ^{-}. They are defined as follows [3, 17]: Let $\varphi^{-} \in \Phi^{-}$ and $\varphi^{-}(E)$ be its corresponding function in $\mathcal{H}^2_{+} \cap S\big|_{\mathbb{R}^+}$, i.e., $V_{-}\varphi^{-} = \varphi^{-}(E)$. Then, the complex conjugate, $[\varphi^{-}(E)]^*$, of $\varphi^{-}(E)$ as well as the complex conjugate of the derivatives,

$$\left[\frac{d^k\varphi^{-}(E)}{dE^k}\right]^* \tag{4.4}$$

belong to $\mathcal{H}^2_{-} \cap S\big|_{\mathbb{R}^+}$. We define [3, 17]

$$\langle\varphi^{-}\big|f_k\rangle = \left[\frac{d^k\varphi^{+}(E)}{dE^k}\right]^*_{z_R} \tag{4.5}$$

i.e., the value of the Hardy function on the lower half-plane given by

$$\left[\frac{d^k\varphi^{+}(E)}{dE^k}\right]^* \tag{4.6}$$

at the point z_R. Note that this definition includes the usual one for $k = 0$.

Remark. Here, we use the same notation, $\varphi^{-}(E)$, to denote the function $\varphi^{-}(E) \in \mathcal{H}^2_{+} \cap S\big|_{\mathbb{R}^+}$ and its unique extension of $\mathcal{H}^2_{+} \cap S$.

As we shall see, the formula [3]

$$\langle\varphi^{-}|f_k\rangle = -\frac{k!}{2\pi i}\int_{-\infty}^{\infty}\frac{[\varphi^{+}(E)]^{*}}{(E - z_R)^{k+1}}\,dE \tag{4.7}$$

is precisely the property we need to find the action of A_{T-} on $|f_k\rangle$.

For $\psi^{+} \in \Phi^{+}$, the mapping V_{+} gives a function $V_{+}\psi^{+} = \psi^{+}(E) \in \mathcal{H}^2_{-} \cap S|_{\mathbb{R}^+}$. On ψ^{+}, the functional $|\tilde{f}_k\rangle \in {}^{\times}\Phi^{+}$ gives

$$\langle\psi^{+}|\tilde{f}_k\rangle = \left[\frac{d^k\psi^{-}(E)}{dE^k}\right]^{*}_{z_R^{*}} \tag{4.8}$$

As for $|f_k\rangle$, the following property holds for $|\tilde{f}_k\rangle$:

$$\langle\psi^{+}|\tilde{f}_k\rangle = \frac{k!}{2\pi i}\int_{-\infty}^{\infty}\frac{[\psi^{-}(E)]^{*}}{(E - z_R^{*})^{k+1}}\,dE \tag{4.9}$$

To obtain $A_{T-}|f_k\rangle$, we use the following identity [see (2.6)]:

$$\langle\psi^{+}|A_{T-}|f_k\rangle = \langle f_k|A^{\dagger}_{T-}\psi^{+}\rangle \tag{4.10}$$

In order to compute (4.10), we need to know which is the function in $\mathcal{H}^2_{-} \cap S|_{\mathbb{R}^+}$ corresponding to $A^{\dagger}_{T-}\psi^{+}$ through V_{-}. We first need to recall that $A^{\dagger}_{T-} = A_{T+}$. According to (2.2), we have

$$V_{-}A_{T+}\psi^{+} = V_{-}V^{\dagger}_{-}CV_{+}\psi^{+} = C\psi^{-}(E) = [\psi^{-}(E)]^{*} \in \mathcal{H}^2_{+} \cap S|_{\mathbb{R}^+} \tag{4.11}$$

Let us call $\eta^{+}(E) := [\psi^{-}(E)]^{*}$ and $\eta^{-} = V_{-}^{-1}\eta^{+}(E) \in \Phi^{-}$. The desired function is $\eta^{-}(E)$ and from the definitions, we immediately see that $\eta^{-} = V_{-}A_{T+}\psi^{+}$. Then,

$$\begin{aligned}
\langle f_k|A^{\dagger}_{T-}\psi^{+}\rangle &= \langle f_k|A_{T+}\psi^{+}\rangle = \langle A_{T+}\psi^{+}|f_k\rangle^{*} = \langle\eta^{-}|f_k\rangle^{*} \\
&= \left[-\frac{k!}{2\pi i}\int_{-\infty}^{\infty}\frac{[\eta^{+}(E)]^{*}}{(E - z_R)^{k+1}}\,dE\right]^{*} \\
&= \frac{k!}{2\pi i}\int_{-\infty}^{\infty}\frac{\eta^{+}(E)}{(E - z_R^{*})^{k+1}}\,dE \\
&= \frac{k!}{2\pi i}\int_{-\infty}^{\infty}\frac{[\psi^{-}(E)]^{*}}{(E - z_R^{*})^{k+1}}\,dE \\
&= \langle\psi^{+}|\tilde{f}_k\rangle
\end{aligned} \tag{4.12}$$

After this chain of identities, we obtain

$$\langle\psi^{+}|A_{T-}|f_k\rangle = \langle\psi^{+}|\tilde{f}_k\rangle \qquad \text{for all} \quad \psi^{+} \in \Phi^{+} \tag{4.13}$$

Thus,

$$A_{T-}\big|f_k\rangle = \big|\tilde{f}_k\rangle \tag{4.14}$$

Analogously, we get

$$A_{T+}\big|\tilde{f}_k\rangle = \big|f_k\rangle \tag{4.15}$$

4.2. The Case $\epsilon_T = \epsilon_I = 1$ (*s* Waves)

Everything goes as in the case of single-pole resonances. Therefore, Eqs. (3.31)–(3.34) can be used, replacing f_0 by f_k.

Now, we shall obtain the action of the time reversal operator on the time evolution semigroups. Let us define the evolution semigroups on the duals ${}^{\times}\mathbf{\Phi}^{\pm,r}$ as

$$\mathcal{W}_{\pm}(t) = \mathcal{U}_{\pm} \otimes I = \begin{pmatrix} \mathcal{U}_{\pm}(t) & 0 \\ 0 & \mathcal{U}_{\pm}(t) \end{pmatrix} \tag{4.16}$$

where $t > 0$ for ${}^{\times}\mathbf{\Phi}^{-,r}$ and $\mathcal{U}_{+}(t)$ and $t < 0$ for ${}^{\times}\mathbf{\Phi}^{+,r}$ and $\mathcal{U}_{-}(t)$. This definition is an immediate consequence of the time behavior for the elements of ${}^{\times}\mathbf{\Phi}^{\pm}$ [13] and the definitions of the duals ${}^{\times}\mathbf{\Phi}^{\pm,r}$. After (2.18) and (2.19), we have

$$\mathcal{A}_{T+}\mathcal{W}_{-}(t)\mathcal{A}_{T-} = \mathcal{W}_{+}(-t), \qquad t > 0 \tag{4.17}$$

$$\mathcal{A}_{T-}\mathcal{W}_{+}(t)\mathcal{A}_{T+} = \mathcal{W}_{-}(-t), \qquad t < 0 \tag{4.18}$$

This result is obtained when we choose the minus sign for $\mathcal{A}_{T-}$ as in (3.13). If we do not make this choice, a minus sign appears in the right-hand side of (4.17) and (4.18).

What is the *physical meaning* of the space doubling? In order to explore it, let us consider the dual spaces

$${}^{\times}\mathbf{\Phi}^{+,r=+} = \begin{pmatrix} {}^{\times}\mathbf{\Phi}^{+} \\ 0 \end{pmatrix}; \qquad {}^{\times}\mathbf{\Phi}^{-,r=-} = \begin{pmatrix} 0 \\ {}^{\times}\mathbf{\Phi}^{-} \end{pmatrix} \tag{4.19}$$

The space ${}^{\times}\mathbf{\Phi}^{+,r=+}$ includes all preparable states, which evolve forwardly with time up to $t = 0$. If $F^{++} \in {}^{\times}\mathbf{\Phi}^{+,r=+}$, its time evolution is given by $\mathcal{W}_{+}(t)$ F^{++} with $t < 0$. If we write $F^{++} = \binom{F^{+}}{0}$, we have

$$\mathcal{W}(t)F^{++} = \begin{pmatrix} \mathcal{U}_{+}(t) & 0 \\ 0 & \mathcal{U}_{+}(t) \end{pmatrix}\begin{pmatrix} F^{+} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathcal{U}_{+}F^{+} \\ 0 \end{pmatrix}, \qquad t < 0 \tag{4.20}$$

On the other hand, the space ${}^{\times}\mathbf{\Phi}^{-,r=-}$ includes all preparable states that evolve backward in time after $t = 0$. In this case, if we call F^{--} any element of ${}^{\times}\mathbf{\Phi}^{-,r=-}$, we have that $F^{--} = \binom{0}{F^{-}}$ with $F^{-} \in {}^{\times}\mathbf{\Phi}^{-}$ and

$$\mathcal{W}_-(-t)F^{--} = \begin{pmatrix} \mathcal{U}_-(-t) & 0 \\ 0 & \mathcal{U}_-(-t) \end{pmatrix} \begin{pmatrix} 0 \\ F^- \end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{U}_-(-t)F^- \end{pmatrix}, \qquad t > 0 \tag{4.21}$$

Analogously, $^\times\Phi^{-,r=+}$ includes all observables that evolve forward in time after $t = 0$ and $^\times\Phi^{+,r=-}$ includes all observables that evolve backward in time up to $t = 0$.

Thus, the operator $\mathcal{A}_{T\pm}$ transforms states that evolve $\binom{\text{forward}}{\text{backward}}$ in time into states evolving $\binom{\text{backward}}{\text{forward}}$ in time, and observables that evolve $\binom{\text{backward}}{\text{forward}}$ in time into observables that evolve $\binom{\text{forward}}{\text{backward}}$ in time.

APPENDIX: TIME REVERSAL

Textbooks in quantum mechanics usually define the time reversal operation in the position representation as $C\psi(\mathbf{x}, t) = \psi^*(\mathbf{x}, -t)$, where the star denotes complex conjugation. One has to be careful with this notation to understand what it really means. Following Wigner, time reversal is an operation such that the following operations performed sequentially give the identity:

time displacement by $t \times$ time reversal
$\times$ time displacement by $t \times$ time reversal

If we denote the time reversal operator by C, a possible solution for it would be $C\psi(t) = \psi(-t)$. However, this kind of operation is obviously linear. The need for an antilinear time reversal operation has been nicely shown by Wigner in the following terms: The above operations results in the identity if and only if

time displacement by $t \times$ time inversion
$=$ time inversion $\times$ time displacement by $-t$

Now, let us consider a system for which the Hamiltonian has a complete set of eigenvectors (for instance, the harmonic oscillator, the bound states of the hydrogen atom, or any system formed by the bound states of the Hamiltonian if any). Then, for any state vector φ, one has

$$\varphi = \sum_n a_n \varphi_n, \qquad \text{where} \quad H\varphi_n = E_n \varphi_n \tag{A.1}$$

Then, apply to it the time reversal operation and assume that it is linear. Then, one has

$$C\varphi = \sum_n a_n C\varphi_n \tag{A.2}$$

Since $[H, C] = 0$, $C\varphi_n$ are also eigenvectors of the Hamiltonian with the same eigenvalue E_n. Therefore, displacement by $t = -(-t)$ on $C\varphi$ gives

$$\sum_n a_n e^{-iE_n t} C\varphi_n \tag{A.3}$$

According to the above-mentioned rule, this should be equal to the result of performing two operations: first the displacement by $-t$ on φ,

$$\sum_n a_n e^{iE_n t} \varphi_n \tag{A.4}$$

and then the time reversal operation C. If C were linear, we would have that

$$\sum_n a_n e^{iE_n t} C\varphi_n \tag{A.5}$$

which does not coincide with the expression given by (A.4). However, they will coincide if C is defined as an antilinear operator. (The simplest antilinear operation on a space of complex functions is the complex conjugation.)

Once we have shown that the time reversal operator must be antilinear, let us proceed with our discussion, now in the energy representation. For the sake of simplicity, we can assume that the Hamiltonian has simple spectrum on $\mathbb{R}^+ := [0, \infty)$ [this happens if, for instance, H is of the form $H = p^2/2m + V(|\mathbf{r}|)$ and we consider s waves only]. Then, H is the multiplication operator by the energy E in the energy representation. Pick some pure state $\psi(E)$. Then apply to it, first, time inversion:

$$C\psi(E) = \psi^*(E) \tag{A.6}$$

where we have taken as the time reversal operator C the complex conjugation (as Wigner does). Then, time displacement by t gives

$$e^{-itE}\psi^*(E) \tag{A.7}$$

If we apply time inversion to (A.7), which is now equivalent to performing the complex conjugation operation, we obtain

$$e^{itE}\psi(E) \tag{A.8}$$

Time displacement by $-t$ on (A.8) finally gives

$$e^{-itE}e^{itE}\psi(E) = \psi(E) \tag{A.9}$$

From here, one deduces that

$$C(\psi(E, t)) = C(e^{-itE}\psi(E)) = e^{itE}\psi^*(E) \tag{A.10}$$

Here, $e^{itE}\psi^*(E)$ is the result of applying time displacement by $-t$ to $\psi^*(E)$. Therefore, $e^{itE}\psi^*(E)$ is what should be identified with the $\psi^*(E, -t)$ [or $\psi^*_{-t}(E)$] that appears in the textbooks. (Note that $\psi^*(E, -t) = e^{itE}\psi^*(E) = [e^{-itE}\psi(E)]^* = [\psi(E, t)]^*$). The same can be argued with respect to the coordinate representation, for which time reversal is represented by an antiunitary operator $\tilde{C}$: $\tilde{C}\psi(\mathbf{x}) = \psi^*(\mathbf{x})$. However, in the momentum representation, the time reversal operator C' behaves as $C\varphi(\mathbf{p}) = \varphi^*(-\mathbf{p}) = [\varphi(-\mathbf{p})]^*$, since the time inversion changes $\mathbf{p}$ into $-\mathbf{p}$. Let $\psi(\mathbf{x})$ be a pure state in the coordinate representation; the corresponding state in the momentum representation is given by

$$\hat{\psi}(\mathbf{p}) = \mathcal{F}\psi(\mathbf{p}) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\mathbf{p}\mathbf{x}}\psi(\mathbf{x})\, d\mathbf{x} \tag{A.11}$$

Taking complex conjugation in (A.11), one gets

$$[\hat{\psi}(\mathbf{p})]^* = [\mathcal{F}\psi]^*(\mathbf{p}) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\mathbf{p}\mathbf{x}}\psi^*(\mathbf{x})\, d\mathbf{x} = \widehat{\psi^*}(-\mathbf{p}) \tag{A.12}$$

which gives

$$C'\hat{\psi}(\mathbf{p}) = \widehat{\tilde{C}\psi}(\mathbf{p}) \Rightarrow C' = \mathcal{F}\tilde{C}\mathcal{F}^{-1} \tag{A.13}$$

where $\mathcal{F}$ and the caret denote the Fourier transform. In this sense, we see that the Fourier transform and time reversal commute. We have in general a situation like this. Assume that we are working in an arbitrary representation and that the Hilbert space $\mathcal{H}$ supports this representation. Then, we define the time reversal operator A_T on $\mathcal{H}$ as

$$A_T\phi = U^{-1}\tilde{C}\phi(\mathbf{x}) \tag{A.14}$$

where U is the unitary mapping carrying the space $\mathcal{H}$ into the space of coordinate wave functions [which is in general of the form $L^2(I)$, where I is a measurable subset of $\mathbb{R}^n$ for some $n = 1, 2, \ldots$] and $\phi(\mathbf{x}) = U\phi$. Obviously,

$$A_T = U^{-1}\tilde{C}U \tag{A.15}$$

Now, let us denote $\phi^T := A_T\phi$. Consider now that $\phi(t) := e^{-itH}\phi$ and call H' the Hamiltonian in the coordinate representation, which means that $H' = UHU^{-1}$. One finds

$$\begin{aligned} A_T\phi(t) &= A_Te^{-itH}\phi = U^{-1}\tilde{C}e^{-itH'}\phi(\mathbf{x}) = U^{-1}e^{itH'}\phi^*(\mathbf{x}) \\ &= [U^{-1}e^{itH'}U]U^{-1}\tilde{C}\phi(\mathbf{x}) = e^{itH}A_T\phi = A_T\phi(-t) = \phi^T(-t) \end{aligned} \tag{A.16}$$

Thus, $A_T\phi(t) = \phi^T(-t)$, which generalizes the equation $C\psi(\mathbf{x}, t) = \psi^*(\mathbf{x}, -t)$.

Table I

ϵ_T	ϵ_I	U_P	A_T
$(-1)^{2j}$	$(-1)^{2j}$	1	C
$-(-1)^{2j}$	$(-1)^{2j}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$
$(-1)^{2j}$	$-(-1)^{2j}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$
$-(-1)^{2j}$	$-(-1)^{2j}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$

However, this is not the whole story. As mentioned in the Introduction and used throughout this paper, Wigner [31, 32] realized that, when constructing projective representations of the Poincaré group extended with time inversion and parity, new possibilities for the time reversal operator exist. These new forms for the time reversal operator are not independent of the representation of the parity and imply a doubling of the space supporting the representation (the space of states). We do not discuss this construction here, but instead present a table with the four possibilities. The four choices are characterized by two parameters that also appear among the parameters that characterize the representations of the extended Poincaré group. If U_P is the operator for parity and A_T the time inversion operator and $A_I = U_P A_T$, the parameters ϵ_T and ϵ_I are defined as

$$A_T^2 = \epsilon_T \mathbf{I}; \qquad A_I^2 = \epsilon_I \mathbf{I} \tag{A.17}$$

Table I shows the four choices.

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